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Large Deviations for the 2D Ising Model: A Lower Bound without Cluster Expansions

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We show that a lower large-deviation bound for the block-spin magnetization in the 2D Ising model can be pushed all the way forward toward its correct "Wulff" value for all $\beta > \beta_c$.

KEY WORDS: Large deviations; Ising model; Wulff construction.

1. INTRODUCTION AND STATEMENT OF RESULTS

The classical large-deviation theory for the 2D nearest-neighbor Ising model is well known[8]: Let $\Lambda(L) \subset Z^2$ be a square box of the volume L^2 and let P_Λ^+ denote the Gibbs measure on Λ with plus boundary conditions, that is, the Hamiltonian H_Λ^+ of a spin configuration σ on $\{-1, 1\}^\Lambda$ is given by

$$-H_\Lambda^+(\sigma) = \frac{1}{2} \sum_{\substack{\langle x, y \rangle \\ x, y \in \Lambda}} \sigma_x \sigma_y + \sum_{\substack{\langle x, y \rangle \\ x \in \Lambda,\, y \in \Lambda^c}} \sigma_x$$

where the summation is over all nearest neighbors x and y. The probability distribution of $\sigma \in \{-1, 1\}^\Lambda$ is defined, then, via

$$P_\Lambda^+(\sigma) = \frac{1}{Z_\Lambda^+} e^{-\beta H_\Lambda^+(\sigma)}$$

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where Z_Λ^+ is the corresponding partition function. Define the block-spin magnetization X_Λ on Λ as

$$X_\Lambda = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x$$

where $\sigma_x = \pm 1$ is the value of the spin at the site $x \in \Lambda$. Set

$$p(t) = \lim_{L \to \infty} \frac{1}{L^2} \log E_\Lambda^+ \exp(t\,|\Lambda|\, X_\Lambda)$$

and let $\phi(m) = \sup_t\{tm - p(t)\}$ be the Legendre–Fenchel transform of p. Then the P_Λ^+ probability that X_Λ is around some $m \in R^1$ decays as $e^{-|\Lambda|\phi(m)}$; more precisely, for any closed interval $[a, b] \subset R^1$,

$$\lim_{L \to \infty} \frac{1}{L^2} \log P_\Lambda^+ \{X_\Lambda \in [a, b]\} = -\min_{m \in [a,b]} \phi(m) \tag{1.1}$$

Note that ϕ is a convex function. If the infinite-volume Gibbs measure for the corresponding Ising model is unique, that is, if the inverse temperature β is less than the phase transition threshold β_c, then ϕ is in addition strictly convex and the classical estimate (1.1) is always nontrivial. On the contrary, if $\beta > \beta_c$, then ϕ has a flat portion on the interval $[-m^*, m^*]$, where $m^* = m^*(\beta) > 0$ denotes the spontaneous magnetization. Consequently, in this case (1.1) does not provide us with too much of the information about $m-s$ inside $[-m^*, m^*]$. An explanation of this phenomenon can be found, for example, in ref. 9: any $m \in [-m^*, m^*]$ corresponds to some infinite-volume Gibbs measure, which has zero specific relative entropy with respect to the infinite-volume plus state P^+, $P^+ = \lim P_\Lambda^+$. Thus the exponential decay of probabilities in this region has to be at most of the surface order rather than of the volume one. In other words, in order to get more interesting results in our case one has to study

$$\frac{1}{L} \log P_\Lambda^+ \{X_\Lambda \in [a, b]\} \tag{1.2}$$

whenever $[a, b] \cap [-m^*, m^*] \neq \varnothing$. The investigation in this direction was, to our knowledge, initiated in ref. 14, where nontrivial two-sided bounds on (1.2) were obtained, without, however, specifying precise values and meaning of the corresponding constants. The really remarkable breakthrough came with the (much more extensive in scope) work of ref. 6. The specific problem of large deviations was initially addressed in the announcement in ref. 15, and, to a certain extent, independently completely solved in another very interesting paper.[12] The results of ref. 12 assert that (1.2) tends to a limit as $L \to \infty$ and, moreover, this limit is given by a certain expression involving the minimum of the Wulff functional for the surface tension of the model. Let us introduce some notations to make this precise; we will be rather laconic in doing so, and all missing details can be found either in ref. 6 or in ref. 12; in fact this paper was written in the course of attempts to understand their results better.

(a) The surface tension. We use the same notation for a unit vector n, $n \in S^1$, and for the corresponding angle $\theta \equiv n$,

$$n = (\cos\theta, \sin\theta)$$

The surface tension $F = F(n)$ is a function defined on S^1 which measures the free energy of the $\pm$-interface in the direction orthogonal to n. See refs. 1 and 12 for precise definition and relevant properties. The fact important for us here is that for the 2D Ising model F equals the mass gap of the dual model in the same direction. The dual model is defined on the dual lattice $Z^{2*} = Z^2 + (1/2, 1/2)$ and its inverse temperature β^* is related to β via the Krammer–Wannier relation

$$\tanh \beta^* = e^{-2\beta}$$

We refer to ref. 12 for a comprehensive discussion of duality. Let $\langle\cdot\rangle^f$ be the correlation function of the infinite-volume dual state (which is unique, since $\beta > \beta_c \Rightarrow \beta^* < \beta_c$ and let $\{u_k\}$ be a sequence of dual vertices, $|u_k| \to \infty$, such that

$$\lim_{k\to\infty} \frac{u_k}{|u_k|} = n$$

Then,[12]

$$-\lim_{k\to\infty} \frac{1}{|u_k|} \log\langle\sigma_0 \sigma_{u_k}\rangle^f = F(n)$$

In particular, this means that F satisfies $F(n) = F(-n) = F(n + \pi/2)$ and that the affine extension of F to R^2 is convex or, equivalently, F is a support function of a certain convex body $W_F \subset R^2$,

$$W_F = \{x \mid \sup_{n\in S^1} \langle n, x\rangle - F(n) \leqslant 0\}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in R^2.

(b) The Wulff shape. The convex body defined above is called the Wulff shape. Its boundary $\gamma_F = \partial W_F$ solves the following variational problem:

Let $\mathscr{A}$ be the class of all closed rectifiable curves in R^2 without self-intersections. For $\gamma \in \mathscr{A}$ define the Wulff functional as

$$\mathscr{W}_F(\gamma) = \int_\gamma F(n_s)\, ds$$

where n_s is the unit normal to γ at s. Then,

$$2w_F \stackrel{\text{def}}{=} \min_{\substack{\gamma \in \mathscr{A} \\ \mathrm{vol}(\gamma) = \mathrm{vol}(\gamma_F)}} \mathscr{W}_F(\gamma) = \mathscr{W}_F(\gamma_F)$$

$\mathscr{W}_F$ is invariant under parallel translations. Moreover, it has a nice expression for dilatations, namely

$$\mathscr{W}_F(a\gamma) = a\mathscr{W}_F(\gamma) \tag{1.3}$$

As an immediate consequence of (1.3) we obtain that $a\gamma_F$ solves the dilatated variational problem:

$$\mathscr{W}_F(a\gamma_F) = 2aw_F = \min_{\substack{\gamma \in \mathscr{A} \\ \mathrm{vol}(\gamma) = a^2 \mathrm{vol}(\gamma_F)}} \mathscr{W}_F(\gamma)$$

To get a still better understanding of the quantity w_F, note that by the results of ref. 1, F is smooth. By the convex duality this means[13] that W_F is strictly convex. Therefore, the parametrization $n \to \gamma_F(n)$ is well defined. An easy computation reveals that

$$\mathscr{W}_F(\gamma_F) = \int_{S^1} F(n)(F + F'')(n)\, dn$$

and

$$\mathrm{vol}(\gamma_F) = \tfrac{1}{2} \int_{S^1} F(n)(F + F'')(n)\, dn$$

Thus, w_F is nothing but the volume of the (unnormalized) Wulff droplet.

We are now in a position to state a (somewhat weak) version of the large-deviation results of refs. 6 and 12:

Theorem 1.1. Let m be sufficiently close to m^* and β large enough. Then

$$\lim_{L \to \infty} \frac{1}{L} \log P_\Lambda^+ \{X_\Lambda \leqslant m\} = -2[w_F \alpha(m)]^{1/2} \tag{1.4}$$

where

$$\alpha(m) = \frac{m^* - m}{2m^*}$$

The original estimates of ref. 6 or ref. 12 are more subtle than just a rough logarithmic convergence claimed above. However, our main concern here are the conditions under which the theorem was proved. The restriction on m has to do with the fact that if $(m^* - m)$ is small, then a Wulff droplet of the volume $\alpha(m)$ can be placed inside a unit square, otherwise the influence of the walls of Λ has to be taken into the consideration and the answer will be more cumbersome than the one on the right-hand side of (1.4). Contrary to this, the condition on β to be large is indispensable both for ref. 6 and ref. 12, since low-temperature cluster expansions are substantially employed. On the other hand, it is generally believed that at least for the 2D Ising model the results should remain true for all $\beta > \beta_c$; in particular, large-deviation estimates (1.4) should not depend on a technical possibility to use cluster expansions.

In this paper we undertake a rather modest step in the direction of justifying this belief, namely we give a proof of a lower large-deviation bound with the correct "Wulff" rate without using cluster expansions. The proof relies on a certain analytical property of the surface tension, which holds true for all $\beta > \beta_c$. To be more specific, our core condition is that of the positive stiffness of F, i.e.,

$$\inf_{n \in S^1} (F + F'') \geqslant \alpha > 0 \tag{1.5}$$

Geometrically the assumption above means that the corresponding Wulff shape has the curvature bounded above. In the particular case of the 2D Ising model (1.5) can be obtained from the exact solution.[1,3] The explicit formula for $F'' + F$ may be found in ref. 2.

Our main result is the following:

Theorem 1.2. Let $m \in (-m^*, m^*)$ be close enough to m^* and $\beta > \beta_c$. Then there exists a sequence of numbers $\{R_\Lambda\}$ such that $R_\Lambda \to m$ as L tends to infinity, and

$$\liminf_{L \to \infty} \frac{1}{L} \log P_\Lambda^+ \{X_\Lambda = R_\Lambda\} \geqslant -2[w_F \alpha(m)]^{1/2} \tag{1.6}$$

The construction behind the proof is very similar to the one employed in ref. 12. We consider a Wulff droplet of the relative volume $\alpha(m)$ inside Λ and surround its boundary by a "sausage" of intermediate boxes [that is, of the size L^ν for some $\nu \in (0, 1)$]. Then the fluctuations of the $\pm$-interface inside this sausage will not result in any substantial volume gain, but will be enough to push the estimate to its correct "Wulff" value asserted in the right-hand side of (1.6).

We follow ref. 12 to make an extensive use of duality relations for the 2D Ising model and inherit some of the notations from this paper. Thus Λ^* is set to denote the box on Z^{2*} which is dual to Λ, $\langle \cdot \rangle^f_{\Lambda^*}$ is the correlation function for the Gibbs measure with free boundary conditions for the dual model on Λ^*, and, for any subset M of the dual lattice, Z_M is the corresponding partition function with free boundary conditions on M. Unlike ref. 12, however, our contours and paths are simple geometric objects (closed curves without self-intersections and self-avoiding curves, respectively) composed of dual bonds and rounded corners via the splitting rule introduced in ref. 6, Chapter 3, and also mentioned in the Section 10 of ref. 12. We say that a path γ leads from u to v, $\gamma: u \to v$, if u and v are the two endpoints of γ. The following formula will be important in Section 4 (see ref. 12 for details):

$$\langle \sigma_u \sigma_v \rangle^f_{\Lambda^*} = \sum_{\substack{\gamma: u \to v \\ \text{inside } \Lambda^*}} e^{-2\beta |\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \tag{1.7}$$

The symbol σ will be used to denote the value of spins both on direct and dual lattices, but we hope this will cause no confusion. Finally, we use various correlation inequalities; all relevant details may be found, for example, in a survey article.[16]

The central idea of the proof is to use the positive stiffness assumption (1.5) to show that the precision with which one can approximate $\langle \sigma_u \sigma_v \rangle^f_{\Lambda^*}$ by $\exp[-d(u, v) F(n_{uv})]$ [where $d(u, v)$ is the distance between u and v, and n_{uv} is the unit vector in the direction of the line uv] happens to be enough to conclude that the overhelming contribution to the right-hand side of (1.7) comes from γ inside a certain intermediate box containing u and v.

In Section 2 we use (1.5) to deduce a sort of uniformly sharp triangle inequality, which is shown to be still true for finite-volume correlations $\langle \cdot \rangle^f_{\Lambda^*}$ in Section 3. Section 4 is devoted to the necessary computations in an intermediate box. The proof of Theorem 1.2 is concluded in Section 5. Finally, in Section 6 we briefly discuss some related results about moderate deviations from m^*.

2. SHARP TRIANGLE INEQUALITY

In this section we show that if the surface tension $F = F(n)$ is a smooth function on S^1 and, moreover, the condition (1.5) of positive stiffness holds, then F satisfies a strong form of the triangle inequality.

Lemma 2.1. Let F be smooth and assume that

$$\inf_{n \in S^1} (F + F'') \geqslant \alpha > 0$$

Then for any triangle

$$a n_a = b n_b + c n_c$$

where n_a, n_b, n_c are unit vectors in R^2 and $a, b, c \geqslant 0$, the following inequality holds:

$$bF(n_b) + cF(n_c) - aF(n_a) \geqslant k\alpha(b\,|n_b - n_a|^2 + c\,|n_c - n_a|^2) \tag{2.1}$$

with $k > 0$ being some universal constant.

Proof. Let us elaborate first on some notions from the convex geometry. The Wulff shape W_F is a convex closed subset of R^2. Its support function Φ is given by

$$\Phi(x) = \sup_{y \in W_F} \langle y, x \rangle = |x|\, F\left(\frac{x}{|x|}\right)$$

The differentiability of F implies, therefore, the differentiability of Φ in $R^2 \backslash 0$ and, as mentioned in the Introduction, this means by duality that the boundary $\gamma_F = \partial W_F$ is strictly convex. Thus the parametrization $n \to \gamma_F(n)$ is well defined. Moreover, let $n^\perp$ denote the unit vector which is counterclockwise orthogonal to n, i.e.,

$$(\cos\theta, \sin\theta)^\perp = (-\sin\theta, \cos\theta) = \frac{d}{d\theta}(\cos\theta, \sin\theta)$$

Then it is easy to see that

$$\nabla\Phi(x) = \frac{x}{|x|}\, F\left(\frac{x}{|x|}\right) + |x| \left(\frac{x}{|x|}\right)^\perp F'\left(\frac{x}{|x|}\right)$$

Thus for any two dual points n and $\gamma_F(n)$ the following relations hold:

$$\gamma_F(n) = nF(n) + n^\perp F'(n)$$

and

$$F(n) = \langle n, \gamma_F(n) \rangle = \sup_{m \in S^1} \langle n, \gamma_F(m) \rangle \tag{2.2}$$

Back to the assumptions of the lemma, we see that

$$aF(n_a) = a\langle n_a, \gamma_F(n_a) \rangle = b\langle n_b, \gamma_F(n_a) \rangle + c\langle n_c, \gamma_F(n_a) \rangle$$

or

$$bF(n_b) + cF(n_c) - aF(n_a)$$
$$= b\langle n_b, \gamma_F(n_b) - \gamma_F(n_a)\rangle + c\langle n_c, \gamma_F(n_c) - \gamma_F(n_a)\rangle$$

Therefore it remains to verify the following bound:

Set $G(m) = \langle n, \gamma_F(n) - \gamma_F(m)\rangle$. Then,

$$G(m) \geqslant k\,|n-m|^2 \tag{2.3}$$

It suffices to check (2.3) for m close to n, otherwise the bound is trivial. But,

$$\frac{dG}{dm} = -\left\langle n, \frac{d\gamma_F}{dm}\right\rangle$$

On the other hand, by (2.2),

$$\frac{d\gamma_F}{dm} = (F + F'')\, m^{\perp}$$

Consequently, for m close to n,

$$\frac{dG}{dm} = -(F+F'')\langle n, m^{\perp}\rangle \geqslant k\alpha\,|n-m|$$

and the claim of the lemma follows.

3. INEQUALITIES FOR CORRELATIONS

In order to derive appropriate relations for finite-volume correlations based on the inequality from the previous section we need some additional facts, namely we need to control the approximation of $\exp[-d(u, v)\, F(n_{uv})]$ by $\langle\sigma_u\sigma_v\rangle^f_{\Lambda^*}$ and we have to take care of the situation when $n_a \sim n_b \sim n_c$. The latter task can be readily accomplished by using monotonicity properties of the infinite-volume correlation function $\langle\cdot\rangle^f$.[16] The main statement of this section asserts that if u, v, and z are integer vertices of some admissible triangle of the size L^ν [with $\nu \in (0, 1)$] which lies deep inside the box $\Lambda^* = \Lambda^*(L)$, then for some $\delta > 0$

$$\langle\sigma_u\sigma_v\rangle^f_{\Lambda^*} \geqslant \langle\sigma_u\sigma_z\rangle^f_{\Lambda^*}\, \langle\sigma_z\sigma_v\rangle^f_{\Lambda^*}\, e^{\delta_\epsilon L^\nu} \tag{3.1}$$

as L becomes large.

To be more precise, let us say that a triangle T with sides a, b, and c is *δ-admissible* if:

(i) $\delta \leqslant \min\{a, b, c\} \leqslant \max\{a, b, c\} \leqslant 3$.

(ii) $a \leqslant b$.

(iii) $0 \in \operatorname{int}(T)$.

Let us fix some scale parameter $v \in (0, 1)$ and a displacement radius r, $r < 1$. Then $\mathfrak{I}_\delta = \mathfrak{I}_\delta(L)$ is defined to be a set of all ordered triples $(u, v, z) \in Z^{2*} \times Z^{2*} \times Z^{2*}$ of integer vertices such that the corresponding triangle T can be scaled into a δ-admissible one, i.e., there exists $x \in B_r(0)$ (the ball of radius r around the origin) such that

$$\frac{1}{2L^v}(T - Lx)$$

is δ-admissible, given the side uv is scaled into a and vz into b.

Theorem 3.1. Assume that F satisfies condition (1.5) of positive stiffness. Then $\forall \delta > 0$

$$\frac{1}{\delta} \limsup_{L \to \infty} \max_{\mathfrak{I}_\delta} \frac{1}{L^v} \log \frac{\langle \sigma_u \sigma_z \rangle^f_{\Lambda^*} \langle \sigma_z \sigma_v \rangle^f_{\Lambda^*}}{\langle \sigma_u \sigma_v \rangle^f_{\Lambda^*}} \leqslant -c < 0 \tag{3.2}$$

where $c = c(r, v)$.

Note that once δ is fixed, (3.1) is just a reformulation of the above claim.

The proof of this theorem is splitted into several lemmas. Let us start by showing that once $T = T(u, v, z)$ is far away from the boundary $\partial \Lambda^*$, one can get rid of the subindex Λ^* in (3.2) without changing the conclusion.

Lemma 3.1. Let t and s belong to some $R \in \mathfrak{I}_\delta$. Then,

$$\langle \sigma_t \sigma_s \rangle^f_{\Lambda^*} \leqslant \langle \sigma_t \sigma_s \rangle^f \leqslant \langle \sigma_t \sigma_s \rangle^f_{\Lambda^*}[1 + o(1)]$$

uniformly in all such T. In other words,

$$\lim_{L \to \infty} \sup_{T \in \mathfrak{I}_\delta} \sup_{t, s \in T} \frac{\langle \sigma_t \sigma_s \rangle^f}{\langle \sigma_t \sigma_s \rangle^f_{\Lambda^*}} = 1 \tag{3.3}$$

Proof. It is clear that for all t and s which belong to the same $T \in \mathfrak{I}_\delta$,

$$\langle \sigma_t \sigma_s \rangle^f_{\Lambda^*} \geqslant e^{-cL^v}$$

for some positive $c>0$. A convenient way to figure out the difference between $\langle \sigma_t \sigma_s \rangle^f_{\Lambda^*}$ and $\langle \sigma_t \sigma_s \rangle^f$ is to use an *FK* representation (see ref. 4 for a complete list of relevant definitions and details). Namely, given a box Λ^*, define a joint probability distribution of a spin configuration $\sigma \in \{-1, 1\}^{\Lambda^*}$ and a bond configuration $n \in \{0, 1\}^{B(\Lambda^*)}$, where $B(\Lambda^*)$ is the set of all dual bonds with both ends in Λ^*, by first assigning independent probabilities 1/2 to each spin $\sigma_x = 1$ and p to each (open) bond $n(b) = 1$ and then conditioning the resulting distribution by the event "*Any two neighboring sites x and y with different spins are connected by a closed bond, i.e., if* $\sigma_x \sigma_y = -1$, *then* $n(\langle x, y\rangle) = 0$." The site marginal of the resulting joint distribution is, then, the Ising model on Λ^* with free boundary conditions and $\beta = -\log(1-p)$, whereas the bond marginal is the so-called dependent percolation model or *FK* measure with free boundary conditions on Λ^*. Let us denote the latter (bond) measure as $\mathscr{P}_\Lambda$. Let also $\mathscr{P}$ be an infinite-volume *FK* measure. Then $\mathscr{P}$ dominates $\mathscr{P}_\Lambda$ in the FKG sense. Recall also that

$$\langle \sigma_t \sigma_s \rangle^f = \mathscr{P}(t \leftrightarrow s)$$

and

$$\langle \sigma_t \sigma_s \rangle^f_{\Lambda^*} = \mathscr{P}_\Lambda(t \leftrightarrow s)$$

where the event "$t \leftrightarrow s$" means that the vertices t and s can be connected by a chain of open bonds. Set now $A = \{t \leftrightarrow \partial\Lambda^* \text{ or } s \leftrightarrow \partial\Lambda^*\}$. Then by the FKG inequality

$$\mathscr{P}_\Lambda(t \leftrightarrow s;\, A^c) \geqslant \mathscr{P}(t \leftrightarrow s;\, A^c)$$

Therefore,

$$\begin{aligned}
\langle \sigma_t \sigma_s \rangle^f - \langle \sigma_t \sigma_s \rangle^f_{\Lambda^*} &= \mathscr{P}(t \leftrightarrow s) - \mathscr{P}_\Lambda(t \leftrightarrow s) \\
&\leqslant \mathscr{P}(t \leftrightarrow s) - \mathscr{P}_\Lambda(t \leftrightarrow s;\, A^c) \leqslant \mathscr{P}(t \leftrightarrow s) - \mathscr{P}(t \leftrightarrow s;\, A^c) \\
&= \mathscr{P}(t \leftrightarrow s;\, A) \leqslant \mathscr{P}(A) \leqslant \sum_{z \in \partial\Lambda^*} [\mathscr{P}(t \leftrightarrow z) + \mathscr{P}(s \leftrightarrow z)] \\
&\leqslant 2\, |\partial\Lambda^*| \max_{z \in \partial\Lambda^*} \max\{\langle \sigma_t \sigma_z \rangle^f, \langle \sigma_s \sigma_z \rangle^f\}
\end{aligned}$$

But,

$$\langle \sigma_u \sigma_v \rangle^f \leqslant \exp[-d(u, v) \inf_{n \in S^1} F(n)]$$

Therefore, since $\min\{d(t, z), d(s, z)\} \geqslant (1-r)L$, we have

$$\langle \sigma_t \sigma_s \rangle^f - \langle \sigma_t \sigma_s \rangle^f_{A^*} = o(\langle \sigma_t \sigma_s \rangle^f_{A^*})$$

uniformly in $T \in \mathfrak{I}_\delta$ and hence (3.3) holds.

Lemma 3.2. We have that

$$\langle \sigma_0 \sigma_x \rangle^f \geqslant \exp\{-\Phi(x)[1+o(1)]\}$$

uniformly in $|x| \to \infty$.

Proof. Choose an odd map $x \to [x]$ from R^2 into Z^2 in such a fashion that $[x]$ is a vertex of a unit plaquette containing x and $[x]=0$ for all x in some fixed small ball around the origin. Set

$$f_n(x) = -\frac{1}{n} \log \langle \sigma_0 \sigma_{[nx]} \rangle^f$$

Then for each x fixed,

$$\lim_{n \to \infty} f_n(x) = \Phi(x) = |x| \, F\left(\frac{x}{|x|}\right) \tag{3.4}$$

The assertion is that the above convergence is in fact uniform on $B_1(0)$. Note first that one can find a constant c such that

$$|f_n(x) - f_n(y)| \leqslant f_n(x-y) + c/n \tag{3.5}$$

Indeed, $|[nx]-[ny]-[n(x-y)]| \leqslant 3$. Consequently,

$$\langle \sigma_0 \sigma_{[nx]} \rangle^f \geqslant e^{-c} \langle \sigma_0 \sigma_{[ny]} \rangle^f \langle \sigma_0 \sigma_{[n(x-y)]} \rangle^f$$

with $e^{-c} = \min_{|t|=3} \langle \sigma_0 \sigma_t \rangle^f$. The same remains true if the roles of x and y are interchanged. Furthermore, let us show that for each $\varepsilon > 0$ there exists a constant $k = k(\varepsilon)$ such that

$$f_n(x) \leqslant k\,|x| + \varepsilon \tag{3.6}$$

for n large enough. In order to do this, just observe that

$$f_{lr+m}(x) \leqslant \frac{lr}{lr+m} f_r(x) + \frac{m}{lr+m} f_m(x) + \frac{l+1}{lr+m} c$$

Fix r large enough to ensure that $c/r < \varepsilon/2$. Since $f_r = 0$ on a small ball around the origin,

$$\sup_{B_1} \frac{f_r(x)}{|x|} \equiv \frac{k}{2} < \infty$$

Thus for l large (3.6) will be satisfied.

Assume now that the conclusion of the lemma is wrong and one can find a sequence $\{x_n\}$, $x_n \in B_1$ such that

$$f_n(x_n) > \Phi(x_n) + \varepsilon$$

Set $x = \lim_{n \to \infty} x_n$ (going to a subsequence if necessary). But, according to (3.5) and (3.6),

$$f_n(x) \geqslant f_n(x_n) - k\,|x - x_n| - \varepsilon/2 - c/n$$

for some large (but fixed) constant k and all large enough n. Therefore, since Φ is a continuous function, we obtain that

$$\liminf_{n \to \infty} f_n(x) \geqslant \Phi(x) + \varepsilon/2$$

which contradicts (3.4).

We are now in a position to prove Theorem 3.1. Let $(u, v, z) \in \mathfrak{I}_\delta$. By symmetry it is enough to consider the case when the angle θ between the line vu and the horizontal axis is between zero and $\pi/4$. Let l_1 and l_2 denote two rays emanating from u under the angles $\pi/2$ and $-\pi/4$, respectively (see Fig. 1) and let A be the smallest of the two sectors cut out by those two rays. If $z \in A$, then by monotonicity properties of $\langle \cdot \rangle^f$,[16] $\langle \sigma_v \sigma_z \rangle^f \leqslant \langle \sigma_v \sigma_u \rangle^f$. Consequently,

$$\frac{\langle \sigma_v \sigma_z \rangle^f \langle \sigma_z \sigma_u \rangle^f}{\langle \sigma_v \sigma_u \rangle^f} \leqslant \langle \sigma_z \sigma_u \rangle^f$$

But $d(z, u) \geqslant 2\delta L^\nu$ by the definition of $\mathfrak{I}_\delta$. Thus, in view of Lemma 2, (3.1) holds with $c = 2 \inf_{n \in S^1} F(n)$.

The same argument remains true for all reflections of A with respect to all lattice symmetry axes passing through v. In fact, a similar reasoning shows that if the conclusion of the theorem is to be violated, it will also be violated for some z such that the angle $\angle zvu$ is sharp. But all such z which in addition do not belong to any of the reflections of the set A mentioned above satisfy the following bound:

$$|n_{vz} - n_{vu}| \geqslant k > 0$$

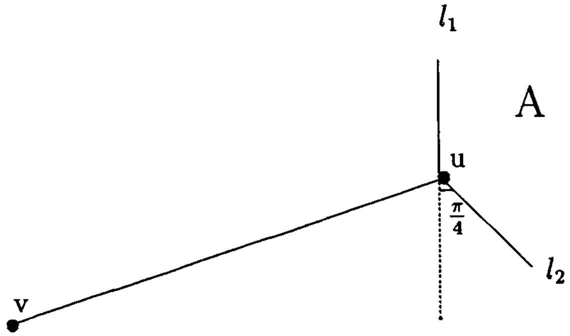

Fig. 1

where k is a fixed independent constant. Therefore, by the sharp triangle inequality of the previous section,

$$d(v, z)\, F(n_{vz}) + d(z, v)\, F(n_{zv}) - d(u, v)\, F(n_{uv}) \geqslant k\delta L^{\nu}$$

It remains to use Lemma 3.2 to justify the shift from F to $\langle \cdot \rangle^f$ and the theorem is completely proved.

4. COMPUTATIONS IN AN INTERMEDIATE BOX

Let us say that two dual vertices u and v form a skeleton pair if:

(i) $L^{\nu} \leqslant d(u, v) \leqslant 2L^{\nu}$.

(ii) Both u and v are deep inside Λ^*, i.e., $d(\{u, v\}, \partial\Lambda^*) \geqslant (1-r)L$ for some fixed $r \in (0, 1)$.

An intermediate box passing through u and v is by the definition an R^2-quadrangle T, $T \subset \Lambda^*$, which possesses the following properties:

(i) u and v lie on the opposite "vertical" sides of T.

(ii) The angles between those two sides and the line uv do not differ too much from $\pi/2$, say they belong to $(\pi/2 - \varepsilon, \pi/2 + \varepsilon)$ with some small ε to be specified later.

(iii) The distance from either u or v to any point on one of the two remaining "horizontal" sides is sandwiched between $2L^{\nu}$ and $3L^{\nu}$.

Given an intermediate box T, we are going to estimate the quantity

$$\sum_{\substack{\gamma:\, u \to v \\ \text{inside } T}} e^{-2\beta|\gamma|} \langle \chi_{\gamma}^{+} \rangle_{\Lambda}^{+} \tag{4.1}$$

where χ_γ^+ is the indicator function of the event that all the spins at sites adjacent to γ are up. Similarly, let χ_γ^- to denote the event that all the spins at sites which harbor an edge intersected by γ are down. The following simple observation will be useful later.

Lemma 4.1. We have

$$\langle \chi_\gamma^+ \rangle_\Lambda^+ \leqslant \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \leqslant 2 \langle \chi_\gamma^+ \rangle_\Lambda^+$$

Proof. The first inequality is obvious. As far as the second one is considered,

$$\frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} = \langle \chi_\gamma^+ \rangle_\Lambda^+ + \langle \chi_\gamma^- \rangle_\Lambda^+ \leqslant 2 \langle \chi_\gamma^+ \rangle_\Lambda^+$$

where the equality follows by duality considerations[12] and the inequality is implied by the first GKS inequality.[15]

Our main import of the lemma above is the possibility to compare $Z_{\Lambda^* \setminus \gamma}$ for different γ.

The main result of this section yields the following bound on the sum (4.1):

Theorem 4.1. There exist two positive constants $c, k > 0$, such that $\forall \delta > 0$,

$$\sum_{\substack{\gamma: u \to v \\ \text{inside } T}} e^{-2\beta |\gamma|} \langle \chi_\gamma^+ \rangle_\Lambda^+ \geqslant c e^{-k\delta L^\nu} \langle \sigma_u \sigma_v \rangle_{\Lambda^*}^f \tag{4.2}$$

simultaneously for all skeleton pairs u and v and all intermediate boxes T passing through $\{u, v\}$, provided only that L is large enough.

Given a skeleton pair $\{u, v\}$, let Γ_{uv} be a fixed shortest path of dual bounds connecting u and v. Pick a vertex z on Γ_{uv} with

$$\delta L^\nu \leqslant d(u, z) \leqslant \delta L^\nu + 1$$

and a vertex w on Γ_{uv} with

$$\delta L^\nu \leqslant d(w, v) \leqslant \delta L^\nu + 1$$

Lemma 4.2. There exist positive constants c and k such that

$$\sum_{\substack{\gamma: u \to v \\ \text{inside } T}} e^{-2\beta |\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \geqslant c e^{-k\delta L^\nu} \sum_{\substack{\gamma: z \to w \\ \text{inside } T}} e^{-2\beta |\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}}$$

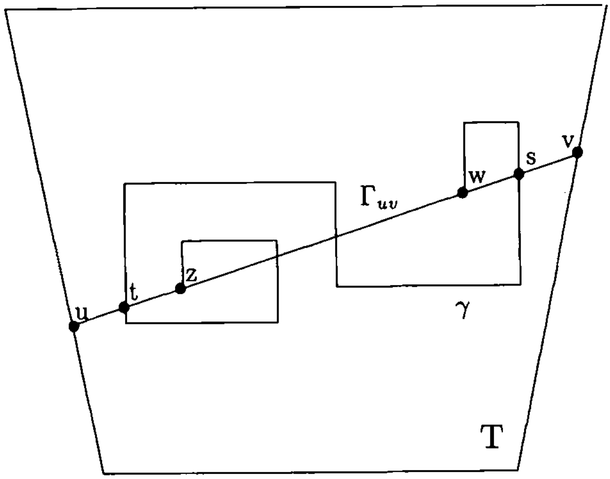

Fig. 2

Proof. Let γ be a path, $\gamma: z \to w$ inside T. We split γ into three pieces, $\gamma = \gamma_{zt} \cup \gamma_{ts} \cup \gamma_{sw}$, where t and s are closest to u and v, respectively, points of intersection of γ and Γ_{uv} (see Fig. 2).

Set Γ_{ut} and Γ_{sv} to be the parts of Γ_{uv} leading from u to t and from s to v, respectively. Note that γ_{ts} does not intersect Γ_{ut} and Γ_{sv} in any points other than t and s. In a view of (1.7) we obtain

$$\begin{aligned}
\sum_{\substack{\gamma: z \to w \\ \text{inside } T}} e^{-2\beta|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} &= \sum_{\substack{t \in \Gamma_{uz} \\ s \in \Gamma_{wv}}} \sum\nolimits^* e^{-2\beta(|\gamma_{zt}| + |\gamma_{ts}| + |\gamma_{sw}|)} \frac{Z_{\Lambda^* \setminus \gamma_{zt} \cup \gamma_{ts} \cup \gamma_{sw}}}{Z_{\Lambda^*}} \\
&\leqslant \sum_{\substack{t \in \Gamma_{uz} \\ s \in \Gamma_{wv}}} \langle \sigma_z \sigma_t \rangle^f_{\Lambda^*} \langle \sigma_s \sigma_w \rangle^f_{\Lambda^*} \sum\nolimits^* e^{-2\beta|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \\
&\leqslant \sum_{\substack{t \in \Gamma_{uz} \\ s \in \Gamma_{wv}}} \sum\nolimits^* e^{-2\beta|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}}
\end{aligned}$$

where, as in ref. 12, $\sum^*$ means the summation over all admissible γ, that is, over all splittings of $\gamma: z \to w$ inside T in the first line, and over all γ_{ts} which satisfy

$$\gamma_{ts} \cap (\Gamma_{ut} \cup \Gamma_{sv} \cup \partial T) = \{t, s\} \tag{4.3}$$

in the second and third lines.

The set of all γ which satisfy condition (4.3) for some $t \in \Gamma_{uz}$ and $s \in \Gamma_{wv}$ can be injected into the set of all paths leading from u to v inside T via the following relation:

$$\gamma_{ts} \to \Gamma_{ut} \cup \gamma_{ts} \cup \Gamma_{sv}$$

Let us denote the corresponding injection by G. By Lemma 4.1,

$$\frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \leqslant 2 \left(\frac{e^{\beta} + e^{-\beta}}{e^{-\beta}} \right)^{-2\,|G(\gamma) \setminus \gamma|} \frac{Z_{\Lambda^* \setminus G(\gamma)}}{Z_{\Lambda^*}}$$

Putting all this together, we finally obtain

$$\sum_{\substack{\gamma:\, z \to w \\ \text{inside } T}} e^{-2\beta\,|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} \leqslant 4\delta L^{\nu} \left(\frac{e^{\beta} + e^{-\beta}}{e^{-\beta}} \right)^{2\delta L^{\nu}} \sum_{\substack{\gamma:\, u \to v \\ \text{inside } T}} e^{-2\beta\,|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}}$$

which concludes the proof of the lemma.

Lemma 4.3. Let z, w, and T be as in Lemma 4.2. Then,

$$\sum_{\substack{\gamma:\, z \to w \\ \text{inside } T}} e^{-2\beta\,|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} = \langle \sigma_z \sigma_w \rangle^{f}_{\Lambda^*} \left[1 - o(1) \right]$$

Proof. Let γ, $\gamma: z \to w$, be a path which exits from T and let us split γ into two peaces, $\gamma = \gamma_{zt} \cup \gamma_{tz}$, where t, $t \in \partial T$, is the point where γ hits ∂T for the first time. Then, $\max\{d(z, t), d(t, w)\} > d(z, w)$ by the definition of an intermediate box, provided only that ε is sufficiently small. Let us assume without loss of generality that $d(t, w) > d(z, w)$. Note that in the notations of the previous section $(z, w, t) \in \mathfrak{I}_{\delta/2}$. Therefore, by Theorem 3.1 there exists a constant $c > 0$ such that

$$\langle \sigma_z \sigma_t \rangle^{f}_{\Lambda^*} \langle \sigma_t \sigma_w \rangle^{f}_{\Lambda^*} \leqslant e^{-c\delta L^{\nu}} \langle \sigma_z \sigma_w \rangle^{f}_{\Lambda^*}$$

for L large enough. Consequently,

$$\begin{aligned} \sum_{\substack{\gamma:\, z \to w \\ \gamma \cup \partial T \neq \varnothing}} e^{-2\beta\,|\gamma|} \frac{Z_{\Lambda^* \setminus \gamma}}{Z_{\Lambda^*}} &\leqslant \sum_{t \in \partial T} \sum{}^{*} e^{-2\beta(|\gamma_{zt}| + |\gamma_{tw}|)} \frac{Z_{\Lambda^* \setminus (\gamma_{zt} \cup \gamma_{tw})}}{Z_{\Lambda^*}} \\ &\leqslant 2\,|\partial T|\, e^{-c\delta L^{\nu}} \langle \sigma_z \sigma_w \rangle^{f}_{\Lambda^*} = o(1) \langle \sigma_z \sigma_w \rangle^{f}_{\Lambda^*} \end{aligned}$$

which is precisely what we need.

The estimate on (4.1) asserted in Theorem 4.1 follows now by a direct successive application of Lemmas 4.1–4.3.

5. PROOF OF THE LOWER BOUND

Let $\Gamma_{m,L}$ be the boundary of the Wulff droplet of the volume $\alpha(m) L^2$ centered inside the box $\Lambda = \Lambda(L)$. We start by giving a precise description of how we construct intermediate boxes around $\Gamma_{m,L}$. Given a scale parameter $\nu \in (0, 1)$, let us go counterclockwise around $\Gamma_{m,L}$ and pick a set of points $x_1, x_2,..., x_{N(L)}$ such that each x_i lies on $\Gamma_{m,L}$ and $L^\nu \leqslant d(x_1, x_{i+1}) \leqslant 2L^\nu$ for all i. With each i we associate a neighboring dual vertex u_i, $d(x_i, u_i) < 1$. Thus, in the language of the previous section, (u_i, u_{i+1}) is a skeleton pair for each $i = 1,..., N(L)$ [with the convention $N(L) + 1 = 1$]. It remains only to draw "vertical" sides of the corresponding intermediate boxes. The self-suggesting idea is to make a vertical side through u_i to be parallel to the direction of the normal to $\Gamma_{m,L}$ at x_i. It is not hard to see that if n_i denotes the unit vector in the direction of the corresponding normal, then

$$\max_i |n_i - n_{i+1}| \sim O(L^{\nu-1}) \tag{5.1}$$

Indeed, since $\Gamma_{m,L}$ can be parametrized by n and in a view of the remarks made in Section 2,

$$ds = \left(\frac{\alpha(m)}{w_F}\right)^{1/2} L(F + F'')\, dn$$

where ds is the element of the length along $\Gamma_{m,L}$. Therefore, using the positive stiffness of F, we conclude that

$$\frac{dn}{ds} \leqslant cL^{-1}$$

and (5.1) follows by the integration on intervals of order L^ν.

Note that the above construction implies that the number ε which appears in the definition of an intermediate box can be chosen as small as we wish, provided that L is sufficiently large.

So let $T_1,..., T_{N(L)}$ be intermediate boxes constructed above and set

$$T = \bigcup_1^{N(L)} T_i$$

It is clear that

$$N(L) \leqslant cL^{1-\nu} \tag{5.2}$$

and

$$|T| \leqslant cL^{1+\nu} \tag{5.3}$$

Furthermore, let $\mathbf{P}_L$ denote the polygon through $u_1,..., u_N$. Then,

$$\frac{1}{L}\mathscr{W}_F(\mathbf{P}_L) \to 2[w_F\alpha(m)]^{1/2} \tag{5.4}$$

as L tends to infinity.

Set now E_L to be the event

$$E_L = \{\textit{there is a } \pm\textit{-contour through } u_1,..., u_N \textit{ inside } T\}$$

where a contour γ is called a $\pm$*-contour* if all the spins at the sites adjacent to γ from the outside are pluses and all the spins at the sites adjacent to γ from the inside are minuses.

The proof of Theorem 1.2 comprises the following two lemmas:

Lemma 5.1. For all $\varepsilon > 0$ there exists $L = L(\varepsilon)$ such that $\forall L \geqslant L(\varepsilon)$,

$$P_\Lambda^+\{|X_\Lambda - m| < \varepsilon/E_L\} \geqslant 1/2$$

Lemma 5.2. We have

$$\liminf_{L\to\infty}\frac{1}{L}\log P_\Lambda^+\{E_L\} \geqslant -2[w_F\alpha(m)]^{1/2}$$

Assume for a moment that both lemmas above are true. Pick a sequence ε_n, $\varepsilon_n \to 0$, and set $L_n = L(\varepsilon_n)$ as described in the statement of Lemma 5.1. Then for each $L \in [L_n, L_{n+1})$ there exists an R_Λ,

$$R_\Lambda \in (m - \varepsilon_n, m + \varepsilon_n)$$

which in addition satisfies

$$P_\Lambda^+\{X_\Lambda = R_\Lambda/E_L\} \geqslant \frac{1}{4\varepsilon_n |\Lambda|}$$

Therefore,

$$\liminf_{L\to\infty}\frac{1}{|L|}\log P_\Lambda^+\{X_\Lambda = R_\Lambda\} \geqslant -[2w_F\alpha(m)]^{1/2}$$

and we are home.

Proof of Lemma 5.1. Let γ be a $\pm$-contour inside T. With a slight abuse of notations we continue to refer to the corresponding event as γ and write $\gamma \in E_L$. Then,

$$P_\Lambda^+\{|X_\Lambda - m| \geqslant \varepsilon/E_L\} \leqslant \sup_{\gamma\in E_L} P_\Lambda^+\{|X_\Lambda - m| \geqslant \varepsilon/\gamma\}$$

Next observe that each $\gamma \in E_L$ splits Λ into two disjoint components: the outer component Λ_1 and the inner component Λ_2. Because of (5.3),

$$P_\Lambda^+\{|X_\Lambda - m| \geqslant \varepsilon/\gamma\} \leqslant P_{\Lambda_1}^+\left\{|X_{\Lambda_1} - m^*| \geqslant \frac{\varepsilon}{3}\right\} + P_{\Lambda_2}^+\left\{|X_{\Lambda_2} - m^*| \geqslant \frac{\varepsilon}{3}\right\}$$

for L large, where X_{Λ_1} and X_{Λ_2} are the block-spin magnetizations in Λ_1 and Λ_2, respectively. The assertion of the lemma reduces, thereby, to a rather rough statement that the law of large numbers is still valid in Λ_1 and Λ_2 as if we are dealing with usual rectangles. We proceed with the proof solely in order to make the presentation here more self-contained. Let us tile Λ_1 with disjoint square boxes of the area at least $L^{2\mu}$ each, $\mu \in (0, 1)$, trying to leave behind as little room as possible. Denote those boxes by $M_1, \ldots, M_{k(L)}$. Obviously, $k(L) \leqslant cL^{2(1-\mu)}$ and

$$\frac{1}{L^2}\left|\Lambda_1 \Big\backslash \left(\bigcup M_i\right)\right| \to 0$$

as L tends to infinity. Set X_{M_i} to be the block-spin magnetization in the box M_i. Then,

$$\left\{|X_{\Lambda_1} - m^*| \geqslant \frac{\varepsilon}{3}\right\} \Rightarrow \left\{\exists i\colon |X_{M_i} - m^*| \geqslant \frac{\varepsilon}{6}\right\}$$

and hence

$$P_{\Lambda_1}^+\left\{|X_{\Lambda_1} - m^*| \geqslant \frac{\varepsilon}{3}\right\} \leqslant k(L) \sup_i P_{\Lambda_1}^+\left\{|X_{M_i} - m^*| \geqslant \frac{\varepsilon}{6}\right\}$$

However, the latter quantity is already under control and we can use the large-deviation results of ref. 5 to bound it above by $cL^{2(1-\mu)} \exp\{-c'L^\mu\}$. Therefore,

$$P_{\Lambda_1}^+\left\{|X_{\Lambda_1} - m^*| \geqslant \frac{\varepsilon}{3}\right\} \to 0$$

as L tends to infinity. $P_{\Lambda_1}^+\{|X_{\Lambda_1} - m^*| \geqslant \varepsilon/3\}$ can be worked out along the same lines.

Proof of Lemma 5.2. With the results of Section 4 in hand, the proof of this lemma is almost identical to the proof of the corresponding statement in ref. 12. Indeed,

$$P_\Lambda^+\{E_L\} = \sum_{\gamma \in E_L} P_\Lambda^+\{\gamma\}$$

We restrict our attention to a subset of E_L which consists of all γ such that the piece of γ connecting each skeleton pair (u_i, u_{i+1}) (and which we denote by γ_i from now on) lies entirely inside the corresponding intermediate box T_i. Thus,

$$P_\Lambda^+\{E_L\} \geqslant \sum_{\substack{\gamma_1:\, u_1 \to u_2 \\ \text{inside } T_1}} \cdots \sum_{\substack{\gamma_N:\, u_N \to u_1 \\ \text{inside } T_N}} P_\Lambda^+\left\{\bigcup_1^N \gamma_i\right\}$$

But,

$$P_\Lambda^+\{\gamma\} = e^{-2\beta|\gamma|}\langle \chi_\gamma^+ \rangle_\Lambda^+$$

for each $\pm$-contour γ. Therefore, by the FKG inequality,

$$P_\Lambda^+\{E_L\} \geqslant \prod_1^N \left(\sum_{\substack{\gamma_i:\, u_i \to u_{i+1} \\ \text{inside } T_i}} e^{-2\beta|\gamma_i|}\langle \chi_{\gamma_i}^+ \rangle_\Lambda^+ \right)$$

Each of the multipliers above can be estimated via Theorem 4.1. Consequently we may conclude that $\forall \delta > 0$,

$$\begin{aligned} P_\Lambda^+\{E_L\} &\geqslant (ce^{-k\delta L^\nu})^{N(L)} \prod_1^{N(L)} \langle \sigma_{u_i}\sigma_{u_{i+1}} \rangle_{\Lambda^*}^f \\ &\geqslant ce^{-k'\delta L} \prod_1^{N(L)} \langle \sigma_{u_i}\sigma_{u_{i+1}} \rangle_{\Lambda^*}^f \end{aligned} \tag{5.5}$$

as L becomes large. Furthermore, by Lemma 3.2,

$$\langle \sigma_{u_i}\sigma_{u_{i+1}} \rangle_{\Lambda^*}^f \geqslant \exp\{-d(u_i, u_{i+1})\, F(n_i)[1 + o(1)]\}$$

Thus we may rewrite (5.5) as

$$P_\Lambda^+\{E_L\} \geqslant c\exp(-k'\delta L)\exp\{-\mathscr{W}_F(\mathbf{P}_L)[1 + o(1)]\}$$

for any $\delta > 0$ and L large. The result follows now from (5.4).

6. MODERATE DEVIATIONS FROM m^*

The results of Sections 3 and 4 and the proof of the main theorem in Section 5 can be almost readily adjusted to derive lower bounds for the probabilities of moderate deviations from m^*. Indeed, assume that the sequence of numbers $\{m(\Lambda)\}$ satisfies

$$\lim_{L \to \infty} L^k \left(\frac{m^* - m(\Lambda)}{2m^*} \right) = c > 0 \tag{6.1}$$

for some $k \in (0, 1)$. Let us try to estimate the probability of X_Λ being close to $m(\Lambda)$ on the scale L^{-k}. One way to do so is to compute the probability of the occurrence of the Wulff droplet of the volume

$$L^2\left(\frac{m^* - m(\Lambda)}{2m^*}\right) \sim L^{2-k}c$$

and then to prove that the existence of such a droplet implies the desirable event. The solution to the former problem follows directly from the estimates of Sections 3 and 4 and we obtain that the probability in question is bounded below by

$$\exp\{-2L^{1-k/2}(cw_F)^{1/2}[1 - o(1)]\} \tag{6.2}$$

Note, however, that (6.2) cannot serve as a correct lower bound for all values of k. For example, in the limiting case $k = 1$ the resulting exponent is $L^{1/2}$, which is too large versus L^0 provided by the central limit theorem. In fact, as explained and proved for large values of β in ref. 6, the threshold value of k is $k_t = 2/3$. According to their results, below k_t the main contribution to moderate deviations comes from the Wulff droplet of maximal volume, whereas above k_t moderate deviations from m^* are of the classical Gaussian nature as in the case of the independent random variables. We can now extend their lower bounds as follows:

Theorem 6.1. Let $\beta > \beta_c$, $k \in [0, 2/3)$, and assume that a sequence $\{m(\Lambda)\}$ satisfies (6.1). Then, there exists a sequence of numbers $\{R_\Lambda\}$ such that

$$\lim_{L\to\infty} L^k(R_\Lambda - m(\Lambda)) = 0$$

and

$$\liminf_{L\to\infty} L^{k/2-1} \log P_\Lambda^+\{X_\Lambda = R_\Lambda\} \geqslant -2(cw_F)^{1/2}$$

The only part to be changed in the proof of Theorem 6.1 as compared to the corresponding proof of Theorem 1.2 in Section 5 is the use of the large-deviation results of ref. 5 to verify the analog of Lemma 5.1. Note, however, that the only thing we need at this point is an appropriate form of the law of large numbers. Thus, since $k < 1$, any central-limit-type result will do equally well. CLT for the block-spin magnetization in a pure phase of the Ising model can be found, for example, in ref. 11.

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